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A PHYSICAL DESCRIPTION OF THE RESPONSE OF COUPLED BEAMS

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An analytical method is presented for computing the vibrational response and the net transmitted power of bending wave fields in system consisting of coupled finite beams. The method is based on a wave approach that utilizes the reflection and transmission coefficients of the different beam joints to couple the elements. These reflection and transmission coefficients are those derived by considering the coupling between the corresponding semi-infinite elements. The predicted results are in almost perfect agreement with exact calculations of the detailed response and net transmitted power. The results are valid for frequencies above which the influence of the reflected near fields for each of the beam elements is negligible. The method is demonstrated on different configurations of beams coupled in extension of each other.

1. INTRODUCTION

Determination of the theoretical response of fundamental simple elements such as uncoupled beams or thin plates can normally be achieved without too much difficulty. However, the analysis of a combined system, that is, one built up by joining several elements together, will generally prove to be quite a challenge as the complexity escalates rapidly with the number of elements involved. Because of practical difficulties, some sort of simplification of the problem will usually be required. This can either be done by employing a numerical method for calculation of the exact response, for example, the Finite Element Method (FEM), or an analytical method which seeks only to determine the approximate response of the structure given as coarse levels. Among the latter can be mentioned Statistical Energy Analysis (SEA) [1], the Mean Value Method (MVM) [2] and Power Flow Analysis (PFA) [3]. A frequent difficulty with the establishment of such models is to include an adequate description of the coupling between elements. For example, in SEA the coupling is described by the so-called coupling loss factors. The frequently employed expressions for these coupling loss factors are based on a travelling wave approach and require a relative high modal overlap [4]. In order to further advance the development of analytical methods, it would be beneficial to have a better understanding of the complex coupling between simple elements.

The effect of coupling various elements can be analyzed by studying the expressions for power transmissions which result from applying one of the well known methods such as the use of receptances [5], transmission matrices [6], the impedance coupling method [7] or the series solutions of the uncoupled elements [8]. Some of the simpler structures that

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can be considered are networks: that is, systems consisting of beams subject to bending and/or longitudinal waves [9–11]. However, the results obtained are specific, and it is rather difficult to reduce the information when the goal is to obtain relative simple general statements concerning the power transmission between elements.

In this paper an alternative method is described for calculating the response and transmitted power of bending waves in structures consisting of beams coupled in extension of each other. The method is less straightforward in its approach and less applicable in its scope than, for example, the dynamic stiffness method [12]. However, its purpose is to describe the response and power transmission in a way that gives a better physical insight. The resulting equations will therefore be easier to reduce to relatively simple parameters for how elements influence each mutually. Possible applications can be mentioned: extension of SEA to use in the medium frequency range characterized by few modes [13] and further development of MVM to handle coupling between elements. Because the description given should be as simple as possible and the main frequency ranges of interest are the medium and high frequency ranges, it is assumed that the influence of the near fields for each beam element is negligible.

In reference [6] (chapter III,3) the response of a single uncoupled beam has been deduced from the behaviour of an infinite one. This technique, which also can be termed ray tracing, is here extended further to deal with a structure consisting of beams coupled in extension of each other. The approach utilizes the fact that the response of a beam can be written as the sum of a propagating wave and an infinite number of reflected waves. The coupling between each beam is described by the reflection and transmission coefficients which are found by considering the coupling between semi-infinite elements [6, 14, 15]. The sums of reflected/transmitted waves can be written as geometric series, and reduced to the so-called generalized reflection and transmission coefficients [16] which also have been employed to study disorder in one-dimensional systems [17, 18] and the behaviour of finite systems [15]. Here, this apparently somewhat cumbersome ray tracing approach is shown to reduce to a simple recursive routine that permits determination of the response and net transmitted power in each beam element. The first part of this paper serves as an introduction to the ray-tracing technique and shows how it can be employed to predict the response of uncoupled and coupled beams. In the second part this procedure is extended to a recursive algorithm which permits an easy determination of the response and net transmitted power of each beam of a large beam structure. The last part is concerned with further extension of the method to more general one-dimensional structures.

2. OUTLINE OF THE THEORY

The wave summation approach [2, 6] permits the introduction of reflection and transmission coefficients in the description of finite systems. The reflection and transmission coefficients which are used in the following are those related to wave amplitudes [22]. The time factor $e^{j\omega t}$ is implicitly included in the expressions for displacement and force, and distributed damping is taken into account in the analysis. This implies a complex wavenumber

$$k \approx \sqrt[4]{m'/B}\sqrt{\omega}(1 - j\eta/4), \tag{1}$$

where η is the loss factor, m' is the mass per unit length, B is the flexural rigidity and ω is the angular velocity. The introduction of a complex wavenumber k in the terms $\exp(-jkx)$ and $\exp(-kx)$ causes the propagating wave to decay exponential and the evanescent fields to propagate in the direction opposite to the direction of the decay [6, see pp. 200–201].

2.1. FORCED RESPONSE OF END DRIVEN BEAM

The first step is to derive the response normalized with respect of the force (the mobility) in physical terms; that is, expressed by the reflection coefficients at the boundaries. Consider the uncoupled element being the end driven beam depicted in Figure 1.

The beam has length l and is excited at $x = 0$ with an harmonic point force F_0. The reflection coefficients at the ends are r_0 and r_1 and the corresponding near field coefficients are r_{j0} and r_{j1}. The forced response of the beam can be expressed as the sum of the propagating wave and an infinite number of its reflections. Starting at $x = 0$ with the propagating wave, the displacement response $w(x)$ of the beam can be written as

$$\begin{aligned} w(x) \simeq\ & w_0\{\mathrm{e}^{-\mathrm{j}kx} + \mathrm{e}^{-kx} + r_1\,\mathrm{e}^{-\mathrm{j}k(2l-x)} + r_{\mathrm{j}1}\,\mathrm{e}^{-\mathrm{j}kl}\,\mathrm{e}^{-k(l-x)}\} \\ & + w_0\{r_0 r_1\,\mathrm{e}^{-\mathrm{j}k(2l+x)} + r_1 r_{\mathrm{j}0}\,\mathrm{e}^{-\mathrm{j}k2l}\,\mathrm{e}^{-kx} + r_0 r_1^2\,\mathrm{e}^{-\mathrm{j}k(4l-x)} + r_0 r_1 r_{\mathrm{j}1}\,\mathrm{e}^{-\mathrm{j}k3l}\,\mathrm{e}^{-k(l-x)}\} \\ & + \cdots . \end{aligned} \tag{2}$$

Consider the first line of the right side of equation (2). The term $\mathrm{e}^{-\mathrm{j}kx}$ corresponds to a wave propagating from $x = 0$ towards the right, and e^{-kx} to a near field wave also propagating from $x = 0$ towards the right. The terms $r_1\,\mathrm{e}^{-\mathrm{j}k(2l-x)}$ and $r_{\mathrm{j}1}\,\mathrm{e}^{-\mathrm{j}kl}\,\mathrm{e}^{-k(l-x)}$ are caused by the incident wave $\mathrm{e}^{-\mathrm{j}kx}$ and correspond to a reflected wave and near field wave both propagating towards the left. The second line of the right side of equation (2) corresponds to the reflections caused by the term $r_1\,\mathrm{e}^{-\mathrm{j}k(2l-x)}$, etc. Equation (2) is not exact, because the contributions resulting from the reflections of the near fields at the ends have been omitted. The near fields will in principle extend from one end to the other where they will be reflected partly as a near field and partly as a bending wave. Of course, pure near fields will normally not contribute to the energy transfer; however, they will in the case in which the field consists of mixed evanescent and propagating waves [20, 21]. To omit the contribution originating from the reflected near fields is a permissible approximation for larger values of the Helmholtz number kl, because of the exponential decay of the near fields.

The response expressed as equation (2) can further be approximated by rewriting it as

$$w(x) \simeq w_0\left[\sum_{n=0}^{\infty} (r_0 r_1\,\mathrm{e}^{-\mathrm{j}k2l})^n\right]\{\mathrm{e}^{-\mathrm{j}kx} + A\,\mathrm{e}^{-kx} + r_1\,\mathrm{e}^{-\mathrm{j}k(2l-x)} + r_{\mathrm{j}1}\,\mathrm{e}^{-\mathrm{j}kl}\,\mathrm{e}^{-k(l-x)}\}. \tag{3}$$

The transition from equation (2) to equation (3) is incorrect with regard to the near field at $x = 0$; the amplitude of this near field is therefore temporarily assigned the value A. For higher values of the wavenumber k, the influence of the near field $A\,\mathrm{e}^{-kx}$ will be negligible on the response for positions near the end $x = l$. Thus, with the exception of the lowest frequencies, equation (3) is seen to satisfy the boundary conditions at $x = l$ because the reflection and the near field reflection coefficients were determined to satisfy equation (3) with $A = 0$. The values of A and "w_0 times the infinite sum" can be found

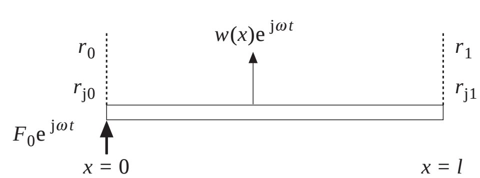


Figure 1. A beam of length l shown with reflection coefficient r and near field reflection coefficients r_{j} for the incident bending wave. The subscripts 0 and 1 refer to position. The beam is driven at the end $x = 0$ by a harmonic point force F_0.

by using the boundary conditions at $x = 0$. Because there is no external moment excitation, this gives

$$\partial^2 w(x)/\partial x^2|_{x=0} = 0 \Rightarrow A \simeq 1 + r_1\, \mathrm{e}^{-\mathrm{j}k2l} \qquad \text{for } \mathrm{e}^{-kl} \approx 0. \tag{4}$$

The transverse force equilibrium gives

$$F_0 = B\frac{\partial^3 w(x)}{\partial x^3}\bigg|_{x=0} \Rightarrow F_0 \simeq w_0 k_B^3 B\left[\sum_{n=0}^{\infty} (r_0 r_1\, \mathrm{e}^{-\mathrm{j}k2l})^n\right]\{\mathrm{j} - 1 - r_1(1+\mathrm{j})\, \mathrm{e}^{-\mathrm{j}k2l}\}. \tag{5}$$

The mobility $Y(x, 0)$ can then be obtained simply by forming the ratio of equations (3) and (5), thereby eliminating the infinite sum, and by multiplying the result by $\mathrm{j}\omega$; that is,

$$Y(x, 0) = \frac{\mathrm{j}\omega w(x)}{F_0} \simeq \frac{\mathrm{j}}{m' c_B}\,\frac{\mathrm{e}^{-\mathrm{j}kx} + (1 + r_1\, \mathrm{e}^{-\mathrm{j}k2l})\, \mathrm{e}^{-kx} + r_1\, \mathrm{e}^{-\mathrm{j}k(2l-x)} + r_{\mathrm{j}1}\, \mathrm{e}^{-\mathrm{j}kl}\, \mathrm{e}^{-k(l-x)}}{\mathrm{j} - 1 - r_1(1+\mathrm{j})\, \mathrm{e}^{-\mathrm{j}k2l}}, \tag{6}$$

where c_B is the phase velocity. Thus, the mobility can very instructively be interpreted as consisting of two main factors represented by the numerator and denominator respectively of the second fraction on the right side of equation (6). First, the numerator represents one propagating wave, one reflected wave from the $x = l$ end and the near fields at the ends. Second, the denominator represents the infinite number of reflections that build up resonances when wavefronts return in phase, a phenomenon referred to as wave-train closure [6]. Of course, the resonances build up at frequencies for which the denominator assumes its minimum values. Throughout the derivation of equation (6), it has been assumed that $\mathrm{e}^{-kl} \approx 0$, and equation (6) is thus an approximation valid for, e.g., $kl > \pi$.

In order to illustrate the previous results, consider the beam to be free at $x = l$; the reflection coefficients are then $r_1 = -\mathrm{j}$ and $r_{\mathrm{j}1} = 1 - \mathrm{j}$. Substituting these coefficients in equation (6) yields an approximate expression for a force driven free–free beam in a form similar to that derived in reference [2]. Results computed from the approximate expression are shown in Figure 2 together with exact analytical results.

The approximations made in the derivation of equation (6) neglect the coupling between the near fields' which occurs at low frequencies. As a consequence of these approximations it is noted that a non-existing "phantom" frequency appears. However, above the first natural frequency of the system the approximate solution and the exact solution are seen to coincide as the errors are of order $\exp(-kl)$.

3. SYSTEM CONSISTING OF TWO COUPLED BEAMS

The beam considered in the previous subsection is now connected to a second beam to form a coupled system, as shown in Figure 3.

The response of the first beam span of this system can be determined from the mobility expression, equation (6), simply by replacing the reflection coefficients r_1 and $r_{\mathrm{j}1}$ with an appropriate set of reflection properties, say, R_1 and $R_{\mathrm{j}1}$. These quantities, that also have been employed in [16], will here be called "net reflection" and "net near field reflection" because they correspond to the net influence that this second beam has on the first beam.

The definitions of R_1 and $R_{\mathrm{j}1}$ as well as of the associated "net transmission" T_1 and "net near field transmission" $T_{\mathrm{j}1}$ are indicated in Figure 4(a), which shows that the original beam for this purpose is temporarily assumed to be semi-infinite. Whereas the ordinary reflection coefficients r_{12}, r_{21} and transmission coefficients t_{12}, t_{21} specify the coupling between two semi-infinite systems (see Figure 4(b)), the net reflection R_1 and transmission T_1 describe the coupling between a semi-infinite system and a finite system.

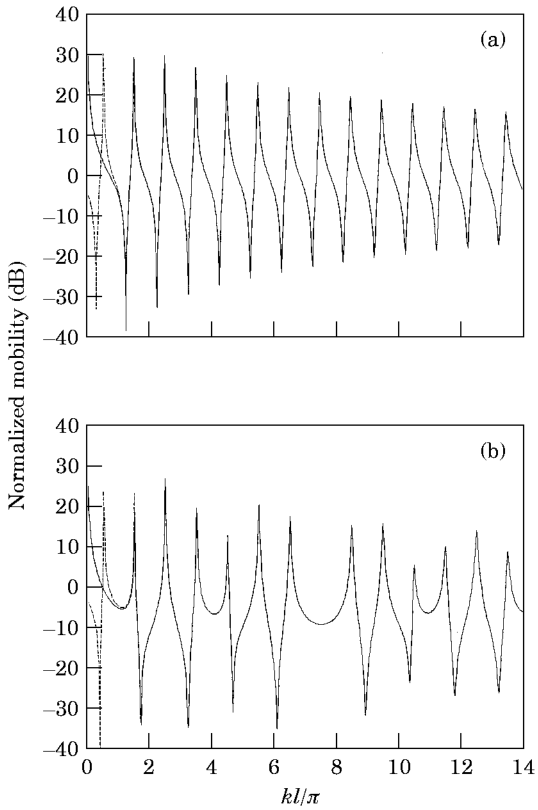


Figure 2. The point ($x/l = 0{\cdot}0$) (a) and transfer ($x/l = 0{\cdot}3$) (b) mobility of a free–free beam driven at the $x = 0$ end, normalized with the characteristic mobility Y_c. ——, Exact solution; - - - - -, approximate solution accordingly to equation (6).

In order to determine the net reflection R_1, consider in Figure 4(a) a wave of unit amplitude coming from $-\infty$. At the intersection, a part of this wave will be reflected, whereas another part will be transmitted to the finite beam. The transmitted part will then gradually be transmitted back to the semi-infinite beam. As seen from the "far field", the net reflection R_1 can be found as the sum of these waves reflected back to $-\infty$, i.e.,

$$R_1 = r_{12} + t_{12}\,\mathrm{e}^{-\mathrm{j}k_2 2l_2} r_2 t_{21} + t_{12}\,\mathrm{e}^{-\mathrm{j}k_2 2l_2} r_2 r_{21}\,\mathrm{e}^{-\mathrm{j}k_2 2l_2} r_2 t_{21} + \cdots . \tag{7}$$

It is seen that R_1 is related to the classical reflection and transmission coefficients. Equation

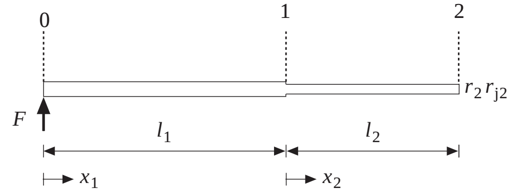


Figure 3. A system of two coupled beams. Subscripts 1 and 2 refer to the first and second beam, respectively. The boundary conditions of the second beam at $x_2 = l_2$ are given by the reflection coefficient r_2 and near field reflection coefficient r_{j2}.

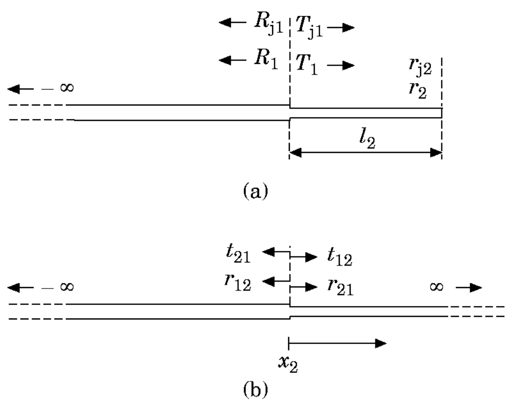


Figure 4. (a) A system for determination of the total reflection R_1 and total transmission T_1. (b) Reflection and transmission coefficients between two semi-infinite elements. The subscripts of, for example, t_{12} indicate transmission from 1 to 2 and those of r_{12} reflection of a wave in 1 impinging at the junction on 2.

(7) can also be expressed as

$$R_1 = r_{12} + t_{12}t_{21}r_2\,\mathrm{e}^{-\mathrm{j}k_2 2l_2}\sum_{n=0}^{\infty}(r_2 r_{21}\,\mathrm{e}^{-\mathrm{j}k_2 2l_2})^n, \tag{8}$$

and because $|r_{21}| < 1$ the infinite sum in equation (8) can be rewritten, yielding the following expression for the net reflection:

$$R_1 = r_{12} + \frac{t_{12}t_{21}r_2}{\mathrm{e}^{\mathrm{j}k_2 2l_2} - r_{21}r_2}, \qquad \begin{Bmatrix} |R_1| = 1 & \text{for} & \eta = 0 \wedge |r_2| = 1 \\ |R_1| < 1 & \text{for} & \eta > 0 \end{Bmatrix}. \tag{9}$$

The net near field reflection R_{j1} can be found by an expression analogous to equation (7), as

$$R_{j1} = r_{j12} + t_{12}t_{j21}r_2/(\mathrm{e}^{\mathrm{j}k_2 2l_2} - r_2 r_{21}). \tag{10}$$

As previously mentioned, the response of the first beam can be obtained by replacing the reflection and near field reflection coefficients in equation (6) with the net reflection and the net near field reflection. This gives the mobility of the end driven beam as

$$Y_1(x_1, 0) \simeq \frac{\mathrm{j}}{m_1' c_{B1}}\,\frac{\mathrm{e}^{-\mathrm{j}k_1x_1} + (1 + R_1\,\mathrm{e}^{-\mathrm{j}k_1 2l_1})\,\mathrm{e}^{-k_1x_1} + R_1\,\mathrm{e}^{-\mathrm{j}k_1(2l_1 - x_1)} + R_{j1}\,\mathrm{e}^{-\mathrm{j}k_1 l_1}\,\mathrm{e}^{-k_1(l_1 - x_1)}}{\mathrm{j} - 1 - R_1(1 + \mathrm{j})\,\mathrm{e}^{-\mathrm{j}k_1 2l_1}}. \tag{11}$$

The response of the second beam in Figure 4(a) must be of the general form

$$w_2(x_2) = A_2\,\mathrm{e}^{-\mathrm{j}k_2x_2} + B_2\,\mathrm{e}^{-k_2x_2} + C_2\,\mathrm{e}^{-\mathrm{j}k_2(l_2 - x_2)} + D_2\,\mathrm{e}^{-k_2(l_2 - x_2)}, \tag{12}$$

which is the solution of the differential equation for bending waves. The quantities A_2, B_2, C_2 and D_2 are now determined under two conditions: first, that the incident wave is of unit amplitude and, second, that the reflections of the near fields are neglected. The quantity A_2, which is the amplitude of the bending wave field propagating in the positive x_2 direction, will be termed the net transmission T_1 when these two conditions are satisfied. The net transmission T_1 can be determined as the sum of waves propagating in the positive

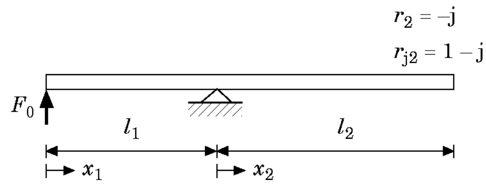


Figure 5. Two beams connected via a simple support. The beams have identical wavenumbers $k_1 = k_2$ and are free at the ends. The first beam is driven by an harmonic point force at $x_1 = 0$.

x_2 direction resulting from an incident wave of unit amplitude; that is,

$$T_1 = t_{12} + t_{12}r_2r_{21}\,e^{-jk_2 2l_2} + t_{12}r_2r_{21}\,e^{-jk_2 2l_2}r_2r_{21}\,e^{-jk_2 2l_2} + \cdots, \tag{13}$$

which yields

$$T_1 = t_{12}\sum_{n=0}^{\infty}(r_2r_{21}\,e^{-jk_2 2l_2})^n = \frac{t_{12}}{1 - r_2r_{21}\,e^{-jk_2 2l_2}}. \tag{14}$$

The quantity B_2 will under the same two conditions correspond to what is termed the net near field transmission T_{j1}. The net near field T_{j1} transmission can be determined from

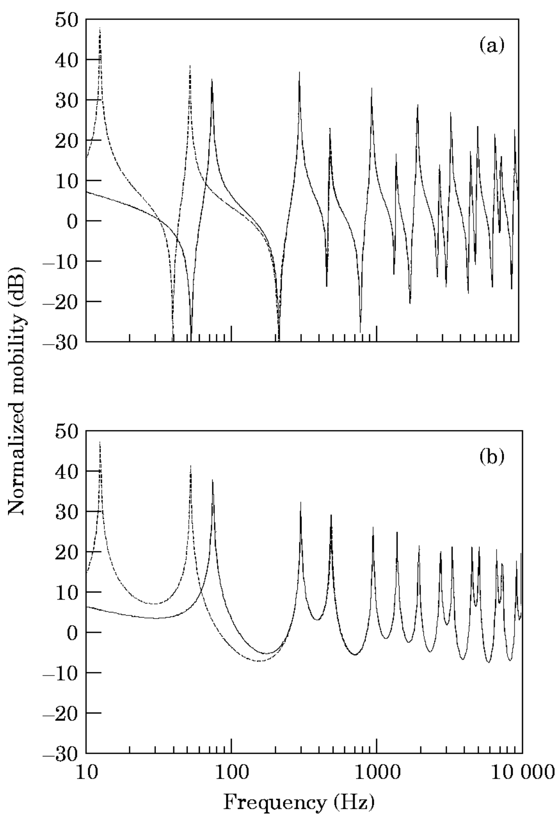


Figure 6. A comparison between the exact and present method solutions illustrated by (a) a point ($x_1 = 0$) and (b) a transfer ($x_2 = l_2$) mobility of the system shown in Figure 5. ——, Exact solution; - - - - -, approximate solution according to equations (11) and (18).

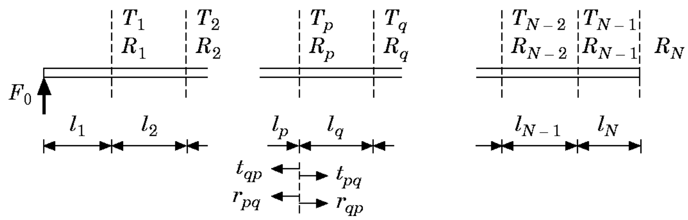


Figure 7. A system of N beams coupled in extension of each other. The coupling between the beams are given by the reflection and transmission coefficients at each joint. Note that $p = q - 1$.

an expression similar to equation (13) as

$$T_{j1} = t_{j12} + t_{12}r_2r_{j21}/(e^{jk_22l_2} - r_2r_{21}). \tag{15}$$

The procedure for determining the remaining two amplitudes C_2 and D_2 is much the same and is hence omitted. The response of the second beam can finally be found as

$$w_2(x_2) \approx T_1\,e^{-jk_2x_2} + T_{j1}\,e^{-k_2x_2} + T_1r_2\,e^{-jk_2(2l_2-x_2)} + T_1r_{j2}\,e^{-jk_2l_2}\,e^{-k_2(l_2-x_2)}. \tag{16}$$

For the finite system in Figure 3, the incident wave will not be of unit amplitude but will consist of an infinite sum of waves as given by the first part of equation (3). This infinite sum of incident waves normalized with the force can be identified from equation (11) as

$$\frac{j\omega w_0}{F_0}\left[\sum_{n=0}^{\infty}(r_0R_1\,e^{-jk_12l_1})^n\right] \approx \frac{j}{m_1'c_{B1}}\frac{e^{-jk_1l_1}}{j-1-R_1(1+j)\,e^{-jk_12l_1}}, \tag{17}$$

where $x_1 = l_1$ has been substituted in the nominator to obtain the correct phase. Multiplying equation (16) with equation (17) gives the mobility of the second beam as

$$Y(x_2, 0) \simeq \frac{j\,e^{-jk_1l_1}}{m_1'c_{B1}}\frac{T_1\,e^{-jk_2x_2} + T_{j1}\,e^{-k_2x_2} + T_1r_2\,e^{-jk_2(2l_2-x_2)} + T_1r_{j2}\,e^{-jk_2l_2}\,e^{-k_2(l_2-x_2)}}{j-1-R_1(1+j)\,e^{-jk_12l_1}}. \tag{18}$$

In order to arrive at equation (18) is has been assumed that k_2l_2 is sufficiently large (e.g., $k_2l_2 > \pi$), so that it is permissible to neglect the influence of the reflections from the near fields.

3.1. NUMERICAL RESULTS

In Figure 5 is shown an example of two beams connected via a simple support. The system consists of two uniform beams with identical wavenumbers, $k_1 = k_2$, connected at $x_1 = l_1$ by a simple support. The first beam is driven at $x_1 = 0$ by a harmonic point force $F_0\,e^{j\omega t}$. In the present case the reflection and transmission coefficients can be found as

$$r_{12} = r_{21} = -(1+j)/2, \qquad t_{12} = t_{21} = (1-j)/2. \tag{19a, b}$$

The beams are assumed to have rectangular cross sections of thickness 0·01 m, width of 0·04 m and lengths of $l_1 = 0.3$ m and $l_2 = 0{\cdot}25$ m. The structural damping corresponds to a loss factor of $\eta = 0{\cdot}01$ and the material is brass. In Figure 6 is shown a comparison between results computed from equations (11) and (18) and exact analytical results. Despite the neglect of the influence of the near fields in the derivations, the results are seen to coincide with the exact solution for higher values of the Helmholtz numbers.

If the response is needed only in the far field, the near field terms can be omitted without disturbing the good agreement. This is particularly interesting, because the near field and far field terms given by this method are determined independently of each other, which is in contrast to the classical method that consists of solving a number of equations: in this case eight equations with eight unknowns. Thus, for estimations in of the response in the far field the terms to be omitted are R_{j1} in equation (11) and T_{j1} and r_{j2} in equation (18).

4. POWER TRANSMISSION

The power transmitted in a system of beams can be estimated from the wave amplitudes in the far fields [23]. If $v_+(x)$ and $v_-(x)$ denote the complex velocity amplitudes of waves propagating in the positive and negative directions in the qth beam span, the time-averaged net power can be estimated by

$$P_{net,q}(x_q) = (B_q k_q^3/\omega)\{|v_{q+}(x_q)|^2 - |v_{q-}(x_q)|^2\}. \tag{20}$$

The wave amplitudes, which depend on damping, can be extracted from equations (11) and (18). This gives the respective powers in the beams of a two-span system as

$$P_{net,1}(x_1) \simeq \frac{|F_0|^2}{m_1' c_{B1}} \frac{|\mathrm{e}^{-\mathrm{j}k_1 x_1}|^2 - |R_1\,\mathrm{e}^{-\mathrm{j}k_1(2l_1 - x_1)}|^2}{|\mathrm{j} - 1 - R_1(1+\mathrm{j})\,\mathrm{e}^{-\mathrm{j}k_1 2l_1}|^2}, \tag{21}$$

$$P_{net,2}(x_2) \simeq \frac{B_2 k_2^3 |F_0\,\mathrm{e}^{-\mathrm{j}k_1 l_1}|^2}{\omega[m_1' c_{B1}]^2} \frac{|T_1\,\mathrm{e}^{-\mathrm{j}k_2 x_2}|^2 - |T_1 r_2\,\mathrm{e}^{-\mathrm{j}k_2(2l_2 - x_2)}|^2}{|\mathrm{j} - 1 - R_1(1+\mathrm{j})\,\mathrm{e}^{-\mathrm{j}k_1 2l_1}|^2}. \tag{22}$$

Equations (21) and (22) give an almost perfect estimation of the net power for $k_q l_q > \pi$, and this condition is often satisfied for frequencies above the lowest natural frequencies of the system. This will be demonstrated in the following for a more complicated system.

4.1. SYSTEM OF MULTIPLE BEAMS

A generalization of the previous results will allow for the prediction of the response and the net transmitted power in each of the beams, which as a whole form a multiple beam system. To understand how this is done, it seems appropriate first to present the equations and then to explain how they are employed. The reader will then realize that this method is a simple extension of the previous results for the two-beam system.

Consider a system consisting of a number of beams (N) coupled in extension of each other, as depicted in Figure 7. The left-hand beam is free at its left end, at which it is also driven by an harmonic point force. The beam at the outmost right end has a boundary condition at its right end represented by the reflection coefficient r_N, which equals the net reflection R_N. The prediction of the response and power transmission in the complete system requires that the net reflections and net transmission are known for each of the joints. These are determined by employing a recursive procedure. Consider, for example, span q, for which the net reflection is R_q. From this the net reflection R_p and transmission

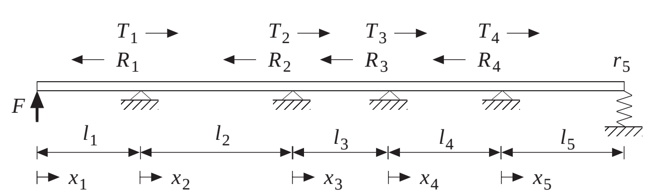


Figure 8. A system of five beams with identical wavenumbers k_q coupled via simple supports. The first beam is free and is driven by an harmonic point force at the end, which the fifth beam is supported by a damped spring at the end.

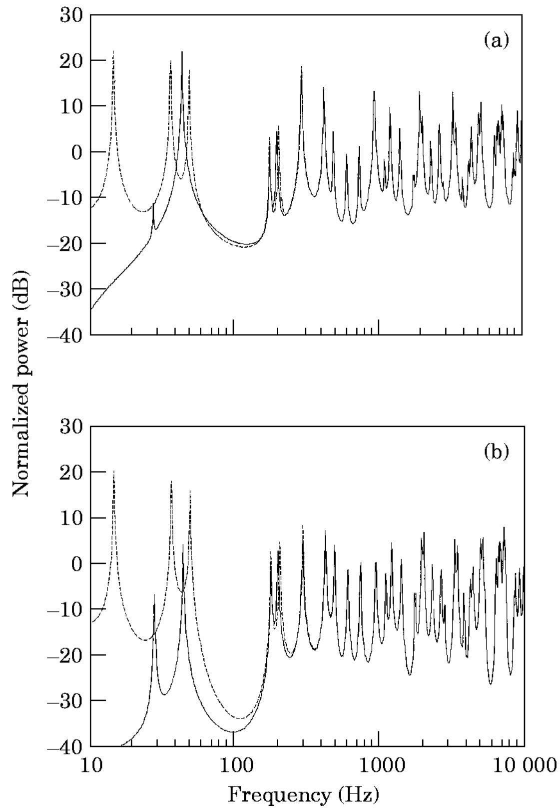


Figure 9. A comparison of net power calculations for a system consisting of five beams with identical wavenumbers, coupled via simple supports as shown in Figure 8. ——, Exact; - - - - -, approximate result obtained with the present method. (a) Beam 1, $x_1 = l_1/2$; (b) beam 3, $x_3 = l_3/2$.

T_p coefficients can be determined, respectively, as

$$R_p = r_{pq} + t_{pq} t_{qp} R_q / (\mathrm{e}^{\mathrm{j}k_q 2l_q} - r_{qp} R_q), \qquad T_p = t_{pq}/(1 - \mathrm{e}^{-\mathrm{j}k_q 2l_q} r_{qp} R_q). \tag{23a, b}$$

The transfer mobility to the far field of beam q is given by

$$Y(x_q, 0) \approx \frac{\mathrm{j}}{m'_1 c_B} K_q \frac{\mathrm{e}^{-\mathrm{j}k_q x_q} + R_q\,\mathrm{e}^{-\mathrm{j}k_q(2l_q - x_q)}}{\mathrm{j} - 1 - R_1(1+\mathrm{j})\,\mathrm{e}^{-\mathrm{j}k_1 2l_1}}, \tag{24}$$

where

$$K_q = \left\{ \begin{array}{ll} 1 & \text{for } q = 1 \\ \prod_{n=1}^{q-1} (\mathrm{e}^{-\mathrm{j}k_n l_n} T_n) & \text{for } q > 1 \end{array} \right\}. \tag{25}$$

From this, the power in span q can be estimated as

$$P_q(x_q) \approx \frac{B_q k_q^3 |F|^2}{\omega [m'_1 c_{B1}]^2} |K_q|^2 \frac{|\mathrm{e}^{-\mathrm{j}k_q x_q}|^2 - |R_q\,\mathrm{e}^{-\mathrm{j}k_q(2l_q - x_q)}|^2}{|\mathrm{j} - 1 - R_1(1+\mathrm{j})\,\mathrm{e}^{-\mathrm{j}k_1 2l_1}|^2}. \tag{26}$$

As an example, the net power for beam 3 becomes

$$P_{net,3}(x_3) \simeq \frac{B_3 k_3^3 |F T_1 T_2\,\mathrm{e}^{-\mathrm{j}k_1 l_1}\,\mathrm{e}^{-\mathrm{j}k_2 l_2}|^2}{\omega [m'_1 c_{B_1}]^2} \frac{|\mathrm{e}^{-\mathrm{j}k_3 x_3}|^2 - |R_3\,\mathrm{e}^{-\mathrm{j}k_3(2l_3 - x_3)}|^2}{|\mathrm{j} - 1 - R_1(1+\mathrm{j})\,\mathrm{e}^{-\mathrm{j}k_1 2l_1}|^2}. \tag{27}$$

To illustrate the use of equations (23)–(26), consider the five-beam system shown in Figure 8. The wavenumber k_q is identical for all five beams, and they are coupled via simple supports. The method for determining the net reflections R_q and the net transmission T_q for each of the joints is of the sequential type. A "backward" determination is applied for determination of R_q and a "forward" determination is used for T_q.

Starting at the right-hand end of the system, the end boundary condition of the fifth beam is given by the reflection coefficient r_5, which equals the net reflection R_5. Equation (23a) is then used to calculate the net reflection R_4, which corresponds to the influence of the fifth beam on the rest of the system. R_4 is then used in equation (23a) to calculate R_3, which corresponds to the influence of the fifth and fourth beams on the rest of the system. R_2 and R_1 are determined in the same manner.

The mobility and power of the first span can then be calculated from R_1 by using equations (24) and (26). In order to deal with the second beam and successive number of beams, the net transmissions T_q are calculated from equation (22b) by using R_{q+1}.

In the Appendix is shown an example of the algorithm for calculation of the net transmitted power in the third beam of the five-beam system shown in Figure 8. Because the net reflection R_1 appears in the denominator, it is always necessary to determine all of the net reflections. However, as the power is only sought for the third beam, it is only necessary to calculate the net transmissions T_1 and T_2.

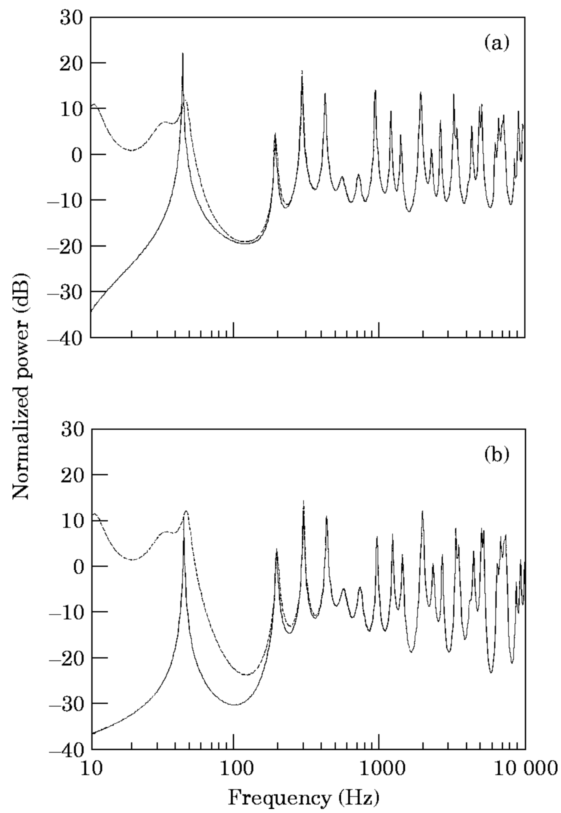
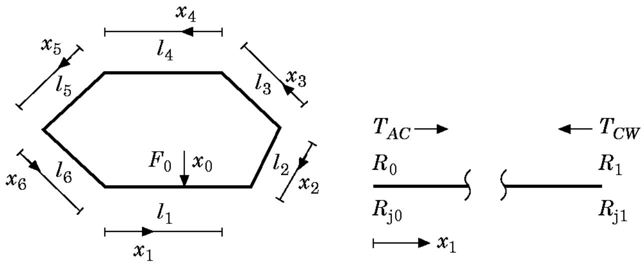


Figure 10. A comparison of net power calculations for a system consisting of five beams with identical wavenumbers, coupled via simple supports. The system is similar to that shown in Figure 8, but with the fifth beam and spring replaced by a semi-infinite beam. ——, Exact; - - - - -, approximate result obtained with the present method. (a) Beam 1, $x_1 = l_1/2$; (b) beam 3, $x_3 = l_3/2$.



(a) (b)

Figure 11. (a) A closed system consisting of $Q = 6$ coupled beams. Beam 1 is driven at x_0 by a harmonic point force F_0. (b) The left and right parts of driven with total reflections R_0, R_1, near fields R_{j0}, R_{j1} and transmitted unit waves in clockwise direction T_{CW} and in anti-clockwise direction T_{AC}.

4.2. NUMERICAL RESULTS

The five-beam system shown in Figure 8 can be used to demonstrate the method. The properties and dimensions of the beam are identical to those used in the previous example (see Figure 5) and the span lengths are $l_1 = 0{\cdot}30$ m, $l_2 = 0{\cdot}25$ m, $l_3 = 0{\cdot}35$ m, $l_4 = 0{\cdot}20$ m and $l_5 = 0{\cdot}40$ m.

Two different end terminations are used. First, the end of the fifth beam is freely supported by a damped spring with a complex stiffness of $20(1 + \mathrm{j}0{\cdot}1)$ N/m and, second, the fifth beam and the spring are replaced by a semi-infinite beam: that is, $l_5 = \infty$.

Typical results for the first and third spans in the two cases are shown in Figures 9 and 10, where these are compared to those of exact computations that have been performed by solving 20 equations with 20 unknowns. Because the agreement at higher frequencies is almost perfect the frequency axis has been chosen to be logarithmic to enable a distinction to be made between the exact and the approximate solutions, which is noticeable only at the low frequencies.

5. SYSTEM DRIVEN IN AN INTERIOR POSITION

The method demonstrated so far has been limited to application to systems consisting of beams where one of beams is driven at its end. This was done in order to facilitate the basic description of the method. The more general case, that is, in which one of beams of the system is driven in what might be termed an "interior" position—that is, a position that is not too close to any of its ends—will be considered in what follows. It will be specified when a position can be said to be an "interior" position.

The system to be examined can be described as a multi-cornered frame; an example of such a system is shown in Figure 11(a). One of the advantages of considering this type of system is that the resulting expressions have a more general character, in that they can be reduced to the analysis of, for example, the situation in which the force position for the system in Figure 8 is changed to an interior position on an arbitrary beam.

5.1. MULTI-CORNERED FRAME

The system that will be considered is made up of Q beams and forms a closed multi-cornered frame with Q corners. An example of such a system is shown in Figure 11(a) for $Q = 6$. The driven beam is excited by a harmonic force F_0 in position x_0. This beam is numbered 1 and the other beams are numbered by incrementing in the anti-clockwise

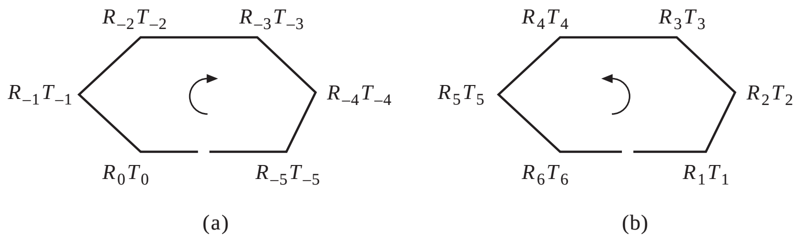


Figure 12. A closed system consisting of $Q = 6$ beams. Total reflections and total transmissions for wave propagation in (a) the clockwise (CW) direction and (b) the anti-clockwise (AC) direction.

direction. As only flexural waves must be present, each corner is thought as being fixed in some manner, thereby preventing the generation of secondary wave types.

The first step consists of deriving expressions for the responses of, respectively, the right-hand side and the left-hand side of the driven beam with the force in position. The two halves of the driven beam are shown in Figure 11(b), where the influence of the other beams is given by the net reflections (R_0 and R_1) and near field reflections (R_{j0} and R_{j1}). The procedure of writing the response as an infinite number of reflected waves is not feasible here; in the introduction to the method, it was employed because it is informative with regard to the approximations being made. For the present purpose it is easier to employ the result from equation (3); that is, to write each of the responses of the two parts of the driven beam as being the sum of a propagating wave and a reflected wave, given by respectively R_0 and R_1, a reflected near field given by respectively R_{j0} and R_{j1}, and a near field originating at the position of the force. However, it is obvious that a part of the wave incident on the right-hand corner of the driven beam will be transmitted through the intermediary beams and will influence the response of the left-hand side of the driven beam. This transmitted part is shown in Figure 11(b) as T_{AC}, where the subscript AC indicates an anti-clockwise direction of propagation. Similarly, the quantity T_{CW} corresponds to the proportion being transmitted from the left-hand side to the right-hand side, where the subscript CW indicates a clockwise direction of propagation. Upon choosing the reference to be the left-hand corner, i.e., $x_1 = 0$, the response of the left-hand side of the driven beam, $w_l(x_1)$, can then be written as

$$w_l(x_1) = w_l\{e^{jk_1x_1} + R_0\, e^{-jk_1x_1} + R_{j0}\, e^{-k_1x_1}\} + w_rT_{AC}\, e^{-jk_1x_1} + A_l\, e^{-k_1(x_0 - x_1)}, \tag{28}$$

where w_l and w_r are the amplitudes of the waves incident on the left- and right-hand joints respectively and A_l is the amplitude of the near field at the force position.

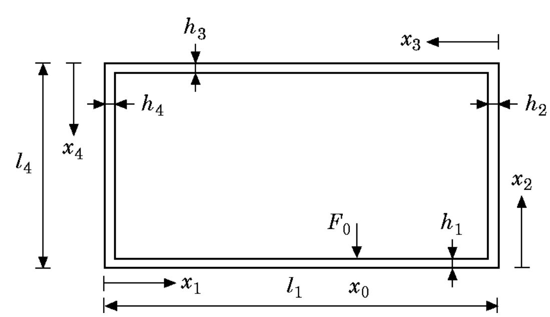


Figure 13. A rectangular frame with the lengths $l_1 = 0{\cdot}5$ m, $l_4 = 0{\cdot}259$ m, width $b = 40$ mm, thickness $h_1 = h_2 = h_4 = 10$ mm and $h_3 = 8$ mm. The material is assumed to be brass. The frame is driven at $x_0 = 0{\cdot}3$ m by an harmonic point force F_0.

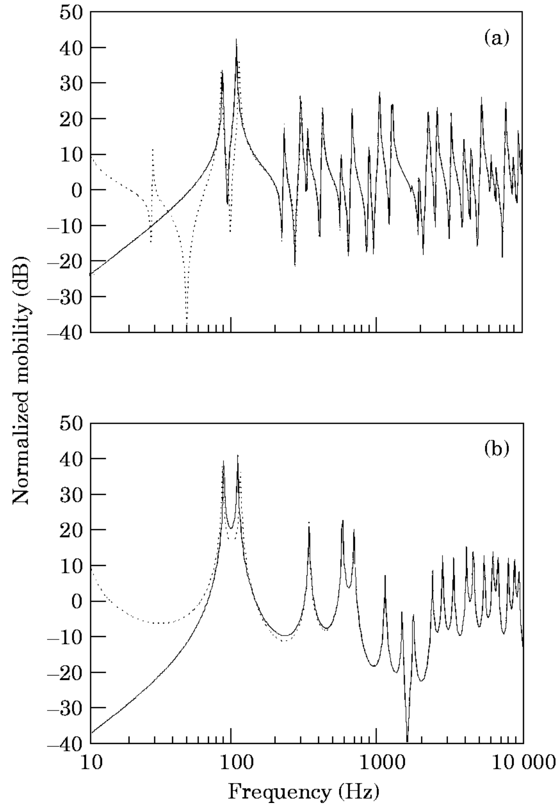


Figure 14. The point ($x_1 = 0{\cdot}3$ m) (a) and transfer ($x_3 = l_3/2$) (b) mobility to the third beam for the rectangular frame subject only to bending waves. ——, Exact solution; - - - - -, approximate solution from equations (33) and (37).

With $x_1 = 0$ still used as reference, the response for the right side $w_r(x_1)$ can be written as

$$w_r(x) = w_r\{\mathrm{e}^{-\mathrm{j}k_1x_1} + R_1\,\mathrm{e}^{-\mathrm{j}k_1(2l_1-x_1)} + R_{\mathrm{j}1}\,\mathrm{e}^{-\mathrm{j}k_1l_1}\,\mathrm{e}^{-k_1(l_1-x_1)} + w_lT_{CW}\,\mathrm{e}^{\mathrm{j}k_1x_1} + A_r\,\mathrm{e}^{-k_1(x_1-x_0)}, \quad (29)$$

where A_r is the amplitude of the near field at the force position.

For the system, two sets of net reflections and the net transmissions need to be determined corresponding to propagation in the clockwise and anti-clockwise direction respectively. This is illustrated in Figures 12(a) and 12(b). The procedure is identical to the one demonstrated previously. For the clockwise direction (see Figure 12(a)), the starting point is the right-hand corner of the driven beam, with the net reflection R_{-5} set equal to the reflection coefficient r_{21}. The net reflections R_{-4} to R_0 can be calculated by employing equation (23a) repeatedly, and once the net reflections have been determined

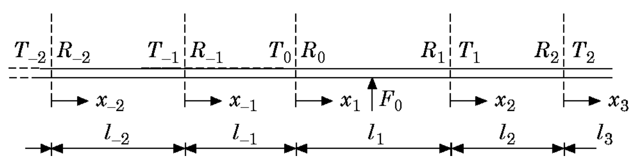


Figure 15. An example of a system consisting of beams coupled in extension of each other. The beams are numbered as shown and the beam with index 1 is driven by an harmonic point force at $x_1 = x_0$.

equation (23b) can be employed to calculate the net transmissions T_0 to T_{-5}. The net reflections R_1 to R_6 and net transmissions T_1 to T_6 for the anti-clockwise direction, shown in Figure 12(b), are determined similarly.

The factors T_{AC} and T_{CW} are determined as follows:

$$T_{AC} = \prod_{n=1}^{Q} T_n \, \mathrm{e}^{-\mathrm{j}k_n l_n}, \qquad T_{CW} = \prod_{n=1}^{Q} T_{(n-Q)} \, \mathrm{e}^{-\mathrm{j}k_n l_n}. \tag{30a, b}$$

The four unknowns w_l, w_r, A_l and A_r can be found from the boundary conditions at the force position, which are

$$w_l(x_0) = w_r(x_0), \qquad \partial w_l(x_1)/\partial x_1|_{x_1 = x_0} = \partial w_r(x_1)/\partial x_1|_{x_1 = x_0}, \tag{31a, b}$$

$$\partial^2 w_l(x_1)/\partial x_1^2|_{x_1 = x_0} = \partial^2 w_r(x_1)/\partial x_1^2|_{x_1 = x_0},$$

$$F = B\{\partial^3 w_r(x_1)/\partial x_1^3|_{x_1 = x_0} - \partial^3 w_l(x_1)/\partial x_1^3|_{x_1 = x_0}\}. \tag{31c, d}$$

The mobility for the left side of the driven beam, i.e., $0 < x_1 \leqslant x_0$, can be found as

$$Y(x_1, x_0)_{left} \approx \frac{w_l\{\mathrm{e}^{\mathrm{j}k_1 x_1} + R_0 \, \mathrm{e}^{-\mathrm{j}k_1 x_1}\} + w_r T_{AW} \, \mathrm{e}^{\mathrm{j}k_1 x_0} + A \, \mathrm{e}^{-k(x_0 - x_1)}}{4m' c_B\{(1 - T_{CW})(1 - T_{AC}) - R_0 R_1 \, \mathrm{e}^{-\mathrm{j}k_1 2 l_1}\}}, \tag{32}$$

and, similarly, the mobility for the right side of the driven beam, $x_0 \leqslant x_1 < l_1$, is found as

$$Y(x_1, x_0)_{right} \approx \frac{w_r\{\mathrm{e}^{-\mathrm{j}k_1 x_1} + R_1 \, \mathrm{e}^{-\mathrm{j}k_1(2l_1 - x_1)}\} + w_l T_{CW} \, \mathrm{e}^{\mathrm{j}k_1 x_0} + A \, \mathrm{e}^{-k_1(x_1 - x_0)}}{4m' c_B\{(1 - T_{CW})(1 - T_{AC}) - R_0 R_1 \, \mathrm{e}^{-\mathrm{j}k_1 2 l_1}\}}, \tag{33}$$

where

$$w_r = (1 - T_{CW}) \, \mathrm{e}^{\mathrm{j}k_1 x_0} + R_0 \, \mathrm{e}^{-\mathrm{j}k_1 x_0}, \qquad w_l = (1 - T_{AC}) \, \mathrm{e}^{-\mathrm{j}k_1 x_0} + R_1 \, \mathrm{e}^{-\mathrm{j}k_1(2l_1 - x_0)} \tag{34}$$

and

$$A = \mathrm{j}\{R_0 R_1 \, \mathrm{e}^{-\mathrm{j}k_1 2 l_1} - (1 - T_{CW})(1 - T_{AC})\}. \tag{35}$$

Equations (32) and (33) are valid for positions not too close to the corners. It has been assumed that

$$\mathrm{e}^{-k x_0} \approx 0, \quad \mathrm{e}^{-k(l - x_0)} \approx 0, \qquad \text{or} \qquad k x_0 > \pi, \quad k(l - x_0) > \pi. \tag{36}$$

The mobilities of the intermediary beams will consist of two contributions. One part will be generated by a wave of amplitude (w_l) travelling in the clockwise direction and the other part by a wave (w_r) travelling in the anti-clockwise direction. Thus the transfer mobility to beam i in the far field can be found as

$$Y(x_i, x_0) \approx \frac{w_r K_{i,AC}\{\mathrm{e}^{-\mathrm{j}k_i x_i} + R_{i,AC} \, \mathrm{e}^{-\mathrm{j}k_i(2l_i - x_i)}\} + w_l K_{i,CW}\{R_{i,CW} \, \mathrm{e}^{-\mathrm{j}k_i x_i} + \mathrm{e}^{\mathrm{j}k_i x_i}\}}{4m' c_B\{(1 - T_{CW})(1 - T_{AC}) - R_0 R_1 \, \mathrm{e}^{-\mathrm{j}k_1 2 l_1}\}}, \tag{37}$$

where

$$K_{i,AC} = \prod_{n=1}^{i-1} \mathrm{e}^{-\mathrm{j}k_n l_n} T_n, \qquad K_{i,CW} = \prod_{n=i}^{Q} \mathrm{e}^{-\mathrm{j}k_n l_n} T_{(n-Q)}. \tag{38a, b}$$

Expressions for the transmitted power can be obtained without difficulty from equations (32), (33) and (37) as described earlier.

5.2. NUMERICAL EXAMPLE

To illustrate the previous findings, consider the rectangular frame depicted in Figure 13. The first beam of the rectangular frame is driven at an off-centre position ($x_0 = 0{\cdot}3$ m). Point and transfer mobilities to the third beam are shown in Figure 14. The outcome of the comparison between the exact calculations and the calculations based on the approximate method is seen to be in accordance with what could be expected from the foregoing sections. Thus, at lower frequencies, where the influence of the reflected near fields are important, the approximate method is incorrect. At higher frequencies, where the influence of the near fields is negligible, the approximate method is in almost perfect agreement with the exact calculations.

5.3. OTHER CONFIGURATIONS

For a system that is not closed, as shown in Figure 15, the response of can be found from the previously derived equations; that is, for the driven beam from equations (32) and (33) by substituting $T_{AC} = T_{CW} = 0$. For the other beams from the response can be calculated from equation (37) by substituting $T_{AC} = T_{CW} = 0$ and using equation (38a) with $K_{i,CW} = 0$ for positions to the right of the driven beam and equation (38b) with $K_{i,AC} = 0$ and $Q = -1$ for positions to the left of the driven beam.

The method can equally well be applied to periodic systems, although the traditional methods for analyzing such systems seems to be more advantageous. The method can, by the use of matrix algebra, be extended to a similar structure subject to coupled bending and longitudinal waves [24].

6. CONCLUSIONS

The method presented predicts the response and net power in a structure consisting of beams coupled in extension of each other and subject to bending waves. The method is based on a "ray tracing" approach which involves the Helmholtz number of each beam and the reflection and transmission coefficients of the joints. The outcome is a simple recursive algorithm which permits calculation of the response and net transmitted power in each element for a structure consisting of beams coupled in extension of each other. This somewhat unconventional approach has the advantage of supplying a more physical description of how coupled elements influence each other, and thereby forms a base for a simplified description of the complex interaction that occurs when elements are joined together. The assumption that the influence of the near fields is negligible is valid only when the Helmholtz number is large for all beam elements. For the cases examined here, in which the Helmholtz numbers are relatively comparable in magnitude, the contribution of near fields to the power transmission is seen to be unimportant at higher frequencies.

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APPENDIX: ALGORITHM FOR CALCULATION OF THE POWER

The algorithm for calculation of the net transmitted power in the third beam of the five beam system shown in Figure 8 is as follows:

//loop start

$f = f_0 + \Delta f n$ // comment: n is incremented for each pass

$k_5 = \sqrt[4]{m_5'/E_5 I_5(1 + \mathrm{j}\eta)}\sqrt{2\pi f}$; $k_4 = \ldots$

// calculation of net reflections

$R_5 = r_5$
$R_4 = r_{45} + \{t_{45}t_{54}R_5\}/\{e^{jk_5 2l_5} - r_{54}R_5\}$
$R_3 = r_{34} + \{t_{34}t_{43}R_4\}/\{e^{jk_4 2l_4} - r_{43}R_4\}$
$R_2 = r_{23} + \{t_{23}t_{32}R_3\}/\{e^{jk_3 2l_3} - r_{32}R_3\}$
$R_1 = r_{12} + \{t_{12}t_{21}R_2\}/\{e^{jk_2 2l_2} - r_{21}R_2\}$

// calculation of net transmissions

$T_1 = t_{12}/\{1 - R_2 r_{21}\, e^{-jk_2 2l_2}\}$
$T_2 = t_{23}/\{1 - R_3 r_{32}\, e^{-jk_3 2l_3}\}$

// calculation of transmitted power

$$P_{net,3}(x_3) \simeq \frac{B_3 k_3^3 |F T_1 T_2\, e^{-jk_1 l_1}\, e^{-jk_2 l_2}|^2}{\omega[m_1' c_{B1}]^2} \frac{|e^{-jk_3 x_3}|^2 - |R_3\, e^{-jk_3(2l_3 - x_3)}|^2}{|j - 1 - R_1(1 + j)\, e^{-jk_1 2l_1}|^2}$$

// loop end.

Here f is the frequency, f_0 is the starting frequency, Δf is the frequency step, n is an integer which is incremented for each pass, k_i is the complex wavenumber for beam i, l_i is the length of beam i, R_i is the net reflection and T_i is the net transmission.